GAUSSIAN PROCESSES
EXERCISE SHEET 0: FRAMEWORK OF PROBABILITY AND SOME BASIC CALCULATIONS

1. Framework (Not Examinable)

The rigorous framework of probability theory will mostly be in the background, pretty much like the construction of real numbers when working with them. Still, it is good to know such a framework exists. So here are a few questions about the general framework of probability.

Exercise 1 (Probability space and random variables)**.** *What is a sample space, a sigma-algebra and a probability measure? Give a definition of a probability space and of a random variable on a probability space. What is a Borel sigma-algebra?*

Exercise 2 (Equality in probability)**.** *What does it mean to be equal in the context of probability theory? Define equality in law, and almost sure equality and find an example of two random variables on the same probability space that are equal in law but not almost surely.*

Exercise 3 (Do Gaussians exist?)**.** *Why does a Gaussian random variable exist? In other words, starting from definitions, construct a probability space and define carefully a real-valued variable whose law is Gaussian.*

Exercise 4 (Independent Gaussians)**.** *For $n \in \mathbb{N}$, construct a probability space with n independent Gaussians.*

Exercise 5 (Different notions of convergence)**.** *Define convergence almost surely, in L^p, in probability, in law. Give examples of random variables that converge in law, but not in probability, and of random variables that converge in probability, but not almost surely.*

2. Basic Calculations and Equivalences

Exercise 6 (Moments for Gaussians)**.** *Let X be a zero mean Gaussian with variance σ^2. For any $n \geq 2$ calculate $\mathbb{E}[X^n]$ in terms of σ^2.*

Exercise 7 (Gaussian tail bounds)**.** *Let X be a standard Gaussian. Prove that for $t > 0$,*

$$\frac{1-e^{-1/2}}{(t+1)\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \;\leq\; \mathbb{P}(X > t) \;\leq\; \frac{1}{t\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right).$$

Exercise 8 (Equivalence of different definitions)**.** *Let $\overline{X}$ be a $\mathbb{R}^n$-valued random variable. Show that the following conditions are equivalent and thus all give a definition of a non-degenerate Gaussian vector:*

1. *There exists an $n \times n$ matrix A of rank n, and a vector $\overline{M}$ such that if $\overline{Y}$ is a vector of n independent standard Gaussians, then $\overline{X} = A\overline{Y} + \overline{M}$ almost surely.*
2. *There is a symmetric positive definite matrix D and a vector $\overline{M}$ such that the density of $\overline{X}$ with respect to the Lebesgue measure $d\overline{x}$ on $\mathbb{R}^n$ is given by*

$$\frac{\det(D)^{1/2}}{(2\pi)^{n/2}} \exp\Big(-\tfrac{1}{2}(\overline{x}-\overline{M})^T D(\overline{x}-\overline{M})\Big).$$

(3) There is a symmetric positive definite matrix C and a vector $\overline{M}$ such that for all vectors $\overline{\lambda} \in \mathbb{R}^n$,

$$\mathbb{E}\big[\exp(\langle \overline{\lambda}, \overline{X} \rangle)\big] = \exp\left(\langle \overline{\lambda}, \overline{M} \rangle + \frac{1}{2} \overline{\lambda}^T C \overline{\lambda} \right).$$

(4) For all non-zero vectors $\overline{\lambda} \in \mathbb{R}^n$, we have that $\langle \overline{\lambda}, \overline{X} \rangle$ is a non-degenerate Gaussian.

Explain how the matrices A, C, D are related to each other.